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# ELLIPTIC PLANAR VECTOR FIELDS WITH DEGENERACIES

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Abstract. This paper deals with the normalization of elliptic vector fields in the plane that degenerate along a simple and closed curve. The associated homogeneous equation $Lu = 0$ is studied and an application to a degenerate Beltrami equation is given.

## 0. Introduction

This paper deals mainly with the normalization and integrability of a class of smooth complex-valued vector fields in the plane. A vector field $L$ in this class will be assumed to be elliptic throughout except on a closed and simple curve $\Sigma$ along which it is supposed to be tangent and such that $L \wedge \overline{L}$ vanishes to a constant order on $\Sigma$. The questions considered here are those of integrability and normalization of $L$ in a tubular neighborhood of $\Sigma$.

We can assume that $\Sigma$ is the circle $\{0\} \times S^1 \subset \mathbb{R} \times S^1$ and that in a neighborhood of $\Sigma$, the vector field $L$ has the expression

$$L_n = \frac{\partial}{\partial \theta} - i r^{n+1} a(r,\theta) \frac{\partial}{\partial r}, \tag{0.1}$$

with $Re(a(0,\theta)) \neq 0$ for every $\theta$. The case $n = 0$ is now well understood (see [CG], [M1], and [M2]). The focus of this paper is then on the case $n \geq 1$.

To achieve a normal form for $L_n$, we construct a $C^\infty$-integral of $L_n$ in a ring $A_\delta = (-\delta,\ \delta) \times S^1$. We use this integral to show that $L_n$ is $C^\infty$-conjugate to the rotation invariant vector field $R_n$ given by

$$R_n = \frac{\partial}{\partial \theta} - i \frac{r^{n+1}}{rP'(r) - nP(r) + \mu r^n} \frac{\partial}{\partial r}\,, \tag{0.2}$$

where $\mu \in \mathbb{C}$ and $P(r)$ is a polynomial with degree at most $n-1$ and such that $Re(P(0)) < 0$. The polynomial $P$ and $\mu$ are uniquely determined by the vector field $L_n$. It follows, in particular, that two distinct vector fields given by (2) cannot be conjugate. A corresponding $C^\infty$-integral of $R_n$ is the function

$$f_n(r,\theta) = \exp\left(\epsilon(r)^n \left(\frac{P(r)}{r^n} + \mu \log|r| + i\theta\right)\right)\,, \tag{0.3}$$

with $\epsilon(r) = \dfrac{r}{|r|}$. Note that since $Re(P(0)) < 0$, $f_n \in C^\infty(A_\delta)$ (for $\delta$ small enough) and that it vanishes to infinite order along $\Sigma$.

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Our motivation for seeking such normal forms is in the subsequent study of the pde's related to the structures defined by $L$. The normal forms allow us to write the equations in such a way that they can be analyzed. Related papers about solvability of vector fields near the characteristic set include [BCH], [BCP], [BgM1], [BgM2], [BhM1], [BhM2], [M1], [M2], [NT], [T1], [T2] and many others (see the extensive list of references contained in [T2]).

The organization of this paper is as follows. In Section 1, we set the preliminaries and recall the main results of [CG], [M1] and [M2] about the case $n = 0$. In Section 2, we construct a unique series that is a formal solution of the equation $L_n u = 0$. The series has the form

$$\frac{P(r)}{r^n} + \mu \log |r| + i\theta + \sum_{j=1}^{\infty} f_j(\theta) r^j \ , \tag{0.4}$$

with $\mu \in \mathbb{C}$, $P$ a polynomial with degree $\leq n-1$, and $f_j \in C^\infty(S^1)$ (or in $C^\omega(S^1)$ when $L$ is real analytic). In general, the series $\sum f_j(\theta) r^j$ appearing in (4) diverges for every $r \neq 0$. This is illustrated by an example in Section 3. In order to construct a nonconstant $C^\infty$ solution of $L_n u = 0$, we study, in Section 4, particular CR equations. Namely, equations of the form

$$\frac{\partial w}{\partial \overline{z}} = \frac{f(z)}{z} \quad \text{and} \quad \frac{\partial w}{\partial \overline{z}} = \mu(z) \frac{\partial w}{\partial z} \tag{0.5}$$

where the coefficients $f(z)$ and $\mu(z)$ are functions of order $o(\log^{-q} \frac{1}{|z|})$ for every $q > 0$. In Section 5, we use the series (4) and results obtained in Section 4 to construct a $C^\infty$-integral for $L_n$. The normal form (3) for $L_n$ is obtained in Section 6. The kernel of the operator $R_n$ is studied in Section 7. We prove that all $C^0$-solutions of $R_n u = 0$, in a neighborhood of the circle $r = 0$, are $C^\infty$ functions. This result does not have a local counterpart. Indeed, for every $p \in \Sigma$, the equation $R_n u = 0$ has continuous solutions defined in a neighborhood of $p$ that are not $C^\infty$. For the distribution solutions, we show that if $u \in \mathcal{D}'(A_\delta)$ solves $R_n u = 0$ and has support in $\Sigma$, then there are constants $c_0, \cdots, c_{n-1}$ such that

$$\langle u, \phi \rangle = \sum_{j=0}^{n-1} c_j \int_0^{2\pi} \frac{\partial^j \phi}{\partial r^j}(0, \theta) d\theta \ , \qquad \forall \phi \in \mathcal{D}(A_\delta). \tag{0.6}$$

In the last section we make use of the normalization of the vector $L_0$ to study a degenerate Beltrami equation.

## 1. Preliminaries and first order case

In this section, we give the preliminary settings and recall the normalization for the case $n = 0$. Let

$$L = a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y} \tag{1.1}$$

be a vector field in $\mathbb{R}^2$. We assume that the coefficients $a$ and $b$, are $C^\infty$ or $C^\omega$ functions, are $\mathbb{C}$-valued and that they do not vanish simultaneously. Let $\overline{L}$ be the complex conjugate vector field

$$\overline{L} = \overline{a} \frac{\partial}{\partial x} + \overline{b} dy. \tag{1.2}$$

The vector field $L$ is said to be elliptic at a point $p$ if $L$ and $\overline{L}$ are independent at $p$. If $L$ is elliptic at each point of an open set $\Omega$, then it is equivalent in $\Omega$ to the CR vector field

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y}.$$

Denote by $\Sigma$ the characteristic set of $L$. That is, the set of points where $L$ fails to be elliptic:

$$\Sigma = \{p \in \mathbb{R}^2;\ L \text{ and } \overline{L} \ \text{ are independent}\}. \tag{1.3}$$

We make the following assumptions

- (H1) $\Sigma$ is a simple and closed curve;
- (H2) $L$ is tangent to $\Sigma$ at each point $p \in \Sigma$;
- (H3) $L \wedge \overline{L}$ vanishes to a constant order $n+1$ along $\Sigma$.

It follows from the local representation of such vector fields (see [T1] or [T2]) that for each given $p \in \Sigma$, there are coordinates $(s, t)$, centered at $p$, such that in a neighborhood of the point $p$, the vector field $L$ is a multiple of

$$\frac{\partial}{\partial t} - is^{n+1}\alpha(s,t)\frac{\partial}{\partial s} \tag{1.4}$$

some real-valued function $\alpha$ satisfying $\alpha(0) \neq 0$. It follows at once, that $L$ satisfies the Nirenberg-Treves condition $(P)$ at each point on $\Sigma$ (see [NT] or [T1] or [T2]). These vector fields are thus locally integrable and locally solvable. In fact, the function $\alpha$ of (1.4) can be assumed to be identically equal to 1 (see [CG]). Thus, a vector field $L$ satisfying hypotheses (H1), (H2), and (H3) can be viewed as follows: in a neighborhood of a point $p \notin \Sigma$, $L$ is equivalent to $\dfrac{\partial}{\partial \overline{z}}$ and in a neighborhood of a point $p \in \Sigma$, $L$ is equivalent to $\dfrac{\partial}{\partial y} - ix^{n+1}\dfrac{\partial}{\partial x}$. These vector fields are therefore well understood when viewed locally. Their global behavior is, however, more complicated. Our aim here is to obtain normal forms for $L$ in a tubular neighborhood of the characteristic set $\Sigma$.

From the assumption (H1), we can assume that $\Sigma$ is a circle, that $L$ is defined in $\mathbb{R} \times S^1$, and that

$$\Sigma = \{0\} \times S^1. \tag{1.5}$$

Let

$$L = \alpha(r,\theta)\frac{\partial}{\partial \theta} + \beta(r,\theta)\frac{\partial}{\partial r}, \tag{1.6}$$

where $(r, \theta)$ are the coordinates in $\mathbb{R} \times S^1$. It follows from hypotheses (H2) and (H3) that there exists $\delta > 0$ such that in the ring

$$A_\delta = (-\delta,\ \delta) \times S^1 \tag{1.7}$$

the vector field $L$ is a multiple of a vector field $L_n$ of the form

$$L_n = \frac{\partial}{\partial \theta} - ir^{n+1}a(r,\theta)\frac{\partial}{\partial r}, \tag{1.8}$$

for some $a \in C^\infty(A_\delta)$ satisfying $Re(a(r,\theta)) \neq 0$ for every $(r,\theta) \in A_\delta$. Without loss of generality, we can assume that

$$Re(a(r,\theta)) > 0 \qquad \forall (r,\theta)\ \in\ A_\delta. \tag{1.9}$$

The linear case $n = 0$ was studied in [M1] and [M2] and the study was completed in [CG]. It is proved in [M1] and [M2] that the complex number

$$\lambda = \frac{1}{2\pi}\int_0^{2\pi} a(0,\theta)d\theta \ \in \mathbb{R}^+ + i\mathbb{R} \tag{1.10}$$

is an invariant that characterizes $L_0$. It is shown in [M2] that if $Im\lambda \neq 0$, then for every $k \in \mathbb{Z}^+$, there exists a $C^k$-diffeomorphism of $A_\delta$ that transforms $L_0$ to a multiple of the vector field

$$T_\lambda = \frac{\partial}{\partial\theta} - ir\lambda\frac{\partial}{\partial r}. \tag{1.11}$$

When $Im\lambda = 0$ (i.e., $\lambda \in \mathbb{R}^+$), it is also proved in [M2] that $L_0$ is equivalent to $T_\lambda$ but only under a $C^{1+\sigma}$-diffeomorphism for some $0 < \sigma < 1$. In [CG], the above result about $C^k$ equivalence is extended to include the case $\lambda \in \mathbb{R}\backslash\mathbb{Q}$.

In the real analytic category, it is proved in [M1] that $L_0$ is $C^\omega$-equivalent to $T_\lambda$, if the equation $L_0 u = 0$ has a nonconstant $C^\omega$ solution. This is equivalent to saying that the holonomy group of $\Sigma$ is periodic. It is proved in [CG] that, when $\lambda \in \mathbb{R}\backslash\mathbb{Q}$ and $\lambda$ satisfies a certain diophantine condition (Bruno condition), the vector field $L_0$ is $C^\omega$-equivalent to $T_\lambda$. It is also proved that there are $C^\omega$ vector fields $L_0$ with $\lambda \in \mathbb{R}$ not satisfying the Bruno condition such that $L_0$ is not $C^\omega$ equivalent to $T_\lambda$.

## 2. Formal integrability

We show that a vector field $L_n$ as in (1.8) has a formal integral. First, we rewrite the vector field in more suitable coordinates.

**Lemma 2.1.** *There is a $C^\infty$ change of coordinates that transforms $L_n$ into a multiple of*

$$\frac{\partial}{\partial\theta} - ir^{n+1}(c_0 + c(r,\theta))\frac{\partial}{\partial r}, \tag{2.1}$$

*where $c_0 = 1 + i\beta \in \mathbb{C}$, and $c(r,\theta) \in C^\infty(A_\delta)$ satisfying $c(0,\theta) \equiv 0$. (The change of coordinates is $C^\omega$ when $L$ is $C^\omega$.)*

*Proof.* With $L_n$ as in (1.8), consider the 1-form $\omega$ given by

$$\omega = dr + ir^{n+1}a(r,\theta)d\theta. \tag{2.2}$$

With our assumption $Rea(r,\theta) > 0$, we have

$$\lambda = \frac{1}{2\pi}\int_0^{2\pi} a(0,\theta)d\theta = a + ib \ \in \ \mathbb{R}^+ + i\mathbb{R}. \tag{2.3}$$

Since $n > 0$, we can replace $r$ by $r_1 = \sqrt[n]{a}r$ in such a way that in the new coordinates $(r_1, \theta)$ we have $Re(\lambda) = 1$. Hence, from now on we can assume that

$$a(r,\theta) = 1 + ib_0 + \gamma_1(\theta) + i\gamma_2(\theta) + ra_1(r,\theta), \tag{2.4}$$

where $b_0 \in \mathbb{R}$, $a_1 \in C^\infty(A_\delta)$, and $\gamma_1,\ \gamma_2 \in C^\infty(S^1)$ are $\mathbb{R}$-valued and have averages on $S^1$ equal to 0, i.e.,

$$\int_0^{2\pi} \gamma_k(\theta)d\theta = 0, \qquad k = 1, 2. \tag{2.5}$$

Consider the new angle $\phi$ defined by

$$\phi(\theta) = \theta + \int_0^\theta \gamma_1(s)ds. \tag{2.6}$$

It follows from the hypothesis on $a$ that $\phi'(\theta) > 0$ and from (2.5) that $\phi(\theta + 2\pi) = \phi(\theta) + 2\pi$. With respect to the coordinates $(r, \phi)$, the form $\omega$ has the expression

$$\omega = dr + ir^{n+1}(1 + i\beta - i\chi(\phi) + O(r))d\phi, \tag{2.7}$$

with $\beta \in \mathbb{R}$, $\chi \in C^\infty(S^1)$, real valued and with zero average on $S^1$. Consider the new variables $(\rho, \phi)$ in $A_\delta$, where

$$\rho = \frac{r}{\sqrt[n]{1 - nr^n m(\phi)}} \quad \text{with} \quad m(\phi) = \int_0^\phi \chi(s)ds. \tag{2.8}$$

A calculation shows that

$$dr + r^{n+1}\chi(\phi) = \frac{d\rho}{\sqrt[n]{1 + n\rho^n m(\phi)}^{n+1}}.$$

In the $(\rho, \phi)$ coordinates, the form $\omega$ is a multiple of

$$d\rho + i\rho^{n+1}(1 + i\beta + O(\rho))d\phi.$$

Consequently, $L_n$ is a multiple of a vector field given by (2.1). □

From now on, we will assume that $L_n$ is given by (2.1). We will show that $L_n$ has a formal first integral. More precisely, we have the following proposition.

**Proposition 2.1.** *Let $L_n$ be as in* (2.1)*. Then there exist unique constants $\mu \in \mathbb{C}$, $\alpha_{-n}, \cdots, \alpha_{-1} \in \mathbb{C}$ and a unique sequence of functions $f_j(\theta) \in C^\infty(S^1)$, $j \in \mathbb{Z}^+$, such that the series*

$$f(r, \theta) = \frac{\alpha_{-n}}{r^n} + \cdots + \frac{\alpha_{-1}}{r} + \mu \log|r| + i\theta + \sum_{j=1}^{\infty} f_j(\theta)r^j \tag{2.9}$$

*solves formally the equation $L_n f = 0$.*

*Remark* 2.1. By a formal solution of the equation $L_n u = 0$, we mean the following. For each $N \in \mathbb{Z}^+$, the function $f_N$ defined by

$$f_N(r, \theta) = \frac{\alpha_{-n}}{r^n} + \cdots + \frac{\alpha_{-1}}{r} + \mu \log|r| + i\theta + \sum_{j=1}^{N} f_j(\theta)r^j \tag{2.10}$$

satisfies $L_n f_N = o(r^N)$.

*Remark* 2.2. When $L_n$ is real analytic, the functions $f_j(\theta) \in C^\omega(S^1)$.

*Proof of Proposition* 2.1. The Taylor expansion of the coefficient of $L_n$ given by (2.1) is

$$T_0(c_0 + c(r, \theta)) = \sum_{j=0}^{\infty} c_j(\theta)r^j, \tag{2.11}$$

with $c_j(\theta) = \dfrac{1}{j!}\dfrac{\partial^j c}{\partial r^j}(0, \theta)$. We write

$$c_j(\theta) = c_j^0 + \gamma_j(\theta), \tag{2.12}$$

where

$$c_j^0 = \frac{1}{2\pi}\int_0^{2\pi} c_j(\theta)d\theta. \tag{2.13}$$

Note that $c_0 = 1 + i\beta$, $\gamma_0 = 0$, and that since for $j \geq 1$, the average of $\gamma_j(\theta)$ on $S^1$ is zero, then

$$\int_0^\theta \gamma_j(s)ds \in C^\infty(S^1). \tag{2.14}$$

In order for the series (2.9) to formally satisfy $L_n u = 0$, we need to have

$$\begin{aligned} i + \sum_{j=1}^{\infty} f_j'(\theta)r^j - ir^{n+1}\sum_{l=0}^{\infty}(c_l^0+\gamma_l(\theta))r^l[\frac{-n\alpha_{-n}}{r^{n+1}} + \cdots + \frac{-\alpha_{-1}}{r^2} \\ + \frac{\mu}{r} + \sum_{j=1}^{\infty} jf_j(\theta)r^{j-1}] = 0. \end{aligned} \tag{2.15}$$

After grouping like terms and equating to zero the coefficient of $r^m$, we obtain the following equations:

$$1 + n\alpha_{-n}c_0 = 0, \quad \text{for } m = 0; \tag{2.16}$$

$$f_m' + \sum_{k=n-m}^{n} ik\alpha_{-k}(c_{m-n+k}^0 + \gamma_{m-n+k}) = 0, \quad \text{for } m = 1, \cdots, n-1; \tag{2.17}$$

$$f_n' - ic_0\mu + \sum_{k=1}^{n} ik\alpha_{-k}(c_k^0 + \gamma_k) = 0, \quad \text{for } m = n; \tag{2.18}$$

$$\begin{aligned} f_m' - i\mu(c_{m-n}^0+\gamma_{m-n}) + \sum_{k=1}^{n}(c_{m-n+k}^0 + \gamma_{m-n+k}) \\ - \sum_{k=1}^{m-n} ikf_k(c_{m-n-k}^0 + \gamma_{m-n-k}) = 0 \quad \text{for } m \geq n+1. \end{aligned} \tag{2.19}$$

It follows from (2.16) that

$$\alpha_{-n} = \frac{-1}{nc_0} \tag{2.20}$$

is uniquely determined. We set $m = 1$ in (2.17) to obtain

$$f'(\theta) = -i((n-1)\alpha_{-(n-1)}c_0 + n\alpha_{-n}(c_1^0 + \gamma_1(\theta))). \tag{2.21}$$

It follows from (2.14) that this equation has a $2\pi$-periodic solution $f_1(\theta)$ if and only if

$$(n-1)c_0\alpha_{-(n-1)} + n\alpha_{-n}c_1^0 = 0. \tag{2.22}$$

This determines $\alpha_{-(n-1)}$ uniquely. For this value of $\alpha_{-(n-1)}$, the function $f_1 \in C^\infty(S^1)$ is determined up to an additive constant of integration $K_1$. By induction, suppose that there are unique constants $\alpha_{-n}, \cdots, \alpha_{n-l}$, with $l < n-1$ so that the differential equations in (2.17) for $m = 1, \cdots, l$ have $2\pi$-periodic solutions $f_1, \cdots, f_l$ that are determined up to additive constants $K_1, \cdots, K_l$. For $m = l+1$, we obtain the equation

$$\begin{aligned} f_{l+1}'(\theta) = -i(n-(l+1))\alpha_{-(n-(l+1))}c_0 \\ - \sum_{k=n-l}^{n} ik\alpha_{-k}(c_{l+1-n+k}^0 + \gamma_{l+1-n+k}(\theta)). \end{aligned} \tag{2.23}$$

It follows from (2.14) that equation (2.23) has a $2\pi$-periodic solution $f_{l+1}$ (determined up to an additive constant) for the unique value of $\alpha_{-(n-(l+1))}$ given by

$$(n-(l+1))c_0\alpha_{-(n-(l+1))} + \sum_{k=n-l}^{n} k\alpha_{-k}c^0_{l+1-n+k} = 0. \tag{2.24}$$

This shows that there are unique constants $\alpha_{-n}, \cdots, \alpha_{-1}$ so that equations (2.17) have $2\pi$-periodic solutions $f_1, \cdots, f_{n-1}$ that are determined up to additive constants.

Now that $\alpha_{-n}, \cdots, \alpha_{-1}$ are determined, there is a unique constant $\mu$ given by

$$-c_0\mu + \sum_{k=1}^{n} k\alpha_{-k}c^0_k = 0 \tag{2.25}$$

so that the equation (2.18) has a $2\pi$-periodic solution $f_n(\theta)$.

For $m = n+1$, equation (2.19) has a $2\pi$-periodic solution $f_{n+1}(\theta)$ if and only if

$$\int_0^{2\pi} \left( \mu c^0_1 + \sum_{k=1}^{n} k\alpha_{-k}c^0_{1+k} - f_1(\theta)c_0 \right) d\theta = 0. \tag{2.26}$$

There is a unique choice of the constant $K_1$ for which equation (2.26) holds. For this choice of $K_1$ (so $f_1$ is now uniquely determined), $f_{n+1}$ is determined up to an additive constant. By induction, suppose the functions $f_1, \cdots, f_l$ are uniquely determined so that equations (2.19) have $2\pi$-periodic solutions $f_{n+1}, \cdots, f_{n+l}$ determined up to additive constants. The equation for $m = n+l+1$ has a $2\pi$-periodic solution $f_{n+l+1}$ if and only if

$$\begin{aligned}(l+1)c_0 \int_0^{2\pi} f_{l+1}(\theta)d\theta = &\int_0^{2\pi} (-\mu c^0_{l+1} + \sum_{k=1}^{n} k\alpha_{-k}c^0_{l+1+k})d\theta \\ &+ \sum_{k=1}^{l} \int_0^{2\pi} kf_k(\theta)(c^0_{l+1-k} + \gamma_{l+1-k}(\theta))d\theta.\end{aligned} \tag{2.27}$$

There is a unique constant $K_{l+1}$ (so $f_{l+1}$ is uniquely determined) for which (2.28) holds. This completes the proof of the proposition. □

## 3. An example

We give an example of a real analytic vector field with $n = 1$ for which the series solution constructed in the previous section diverges for every $r \neq 0$. Consider the vector field

$$L_1 = \frac{\partial}{\partial\theta} - ir^2(1 + re^{i\theta})\frac{\partial}{\partial r}. \tag{3.1}$$

We have the following proposition.

**Proposition 3.1.** *For the vector field $L_1$ of* (3.1)*, the series*

$$f(r,\theta) = \frac{\alpha_{-1}}{r} + \mu\log|r| + i\theta + \sum_{j=1}^{\infty} f_j(\theta)r^j, \tag{3.2}$$

*with* $f_j \in C^\infty(S^1)$, *solves formally* $L_1 u = 0$, *if and only if*

$$\alpha_{-1} = -1, \quad \mu = 0, \quad f_1(\theta) = e^{i\theta} \quad \text{and}$$

$$f_j(\theta) = (j-1)!e^{i\theta} + \sum_{k=2}^{j-1} f_{jk}\, e^{ik\theta}, \quad \text{for } j = 2, 3, \cdots, \tag{3.3}$$

*where* $f_{jk}$ *are constants. Consequently, the series* $\sum_{j=1}^\infty f_j(\theta) r^j$ *diverges for every* $r \neq 0$.

*Proof.* It follows at once from $L_1 u = 0$ that

$$\begin{aligned} i(1+\alpha_{-1}) &+ (f_1' + i\alpha_{-1}\mathrm{e}^{i\theta} - i\mu)r + (f_2' - i\mu\mathrm{e}^{i\theta} - if_1)r^2 \\ &+ \sum_{m\geq 3}^{\infty} \left(f_m' - i(m-1)f_{m-1} - i(m-2)\mathrm{e}^{i\theta} f_{m-2}\right) r^m = 0. \end{aligned} \tag{3.4}$$

Hence $\alpha_{-1} = -1$ and then

$$f_1'(\theta) - i\mathrm{e}^{i\theta} - i\mu = 0 \tag{3.5}$$

has a $2\pi$-periodic solution only when $\mu = 0$. In this case

$$f_1(\theta) = \mathrm{e}^{i\theta} + K_1. \tag{3.6}$$

By equating the coefficient of $r^2$ to 0, we get

$$f_2'(\theta) = if_1(\theta) = i\mathrm{e}^{i\theta} + iK_1. \tag{3.7}$$

This equation has a $2\pi$-periodic solution if $K_1 = 0$ and then

$$f_2(\theta) = \mathrm{e}^{i\theta} + K_2. \tag{3.8}$$

By equating the coefficient of $r^3$ to zero, we see that $f_3$ exists only when $K_2 = 0$ and then

$$f_3(\theta) = 2\mathrm{e}^{i\theta} + \frac{1}{2}\mathrm{e}^{2i\theta} + K_3. \tag{3.9}$$

By induction, suppose that $f_j$ has the expression given in (3.3) for $j = 1, \cdots, m-1$, then it follows from (3.4) that

$$\begin{aligned} f_m'(\theta) &= (m-1)if_{m-1}(\theta) + (m-2)i\mathrm{e}^{i\theta} f_{m-2}(\theta) \\ &= (m-1)!i\mathrm{e}^{i\theta} + \sum_{k=2}^{m-1} d_{mk}\mathrm{e}^{ik\theta} \end{aligned} \tag{3.10}$$

with $d_{mk}$ constants. Expression (3.3) for $f_m$ follows at once.

To complete the proof of the proposition, observe that if $\sum f_j(\theta) r^j$ has positive radius of convergence, then the function

$$M(r) = \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{j=1}^\infty f_j(\theta) r^j\right) \mathrm{e}^{-i\theta} d\theta \tag{3.11}$$

would be real analytic at $r = 0$. But it follows from (3.3) that

$$M(r) = \sum_{j=1}^\infty (j-1)! r^j \tag{3.12}$$

with radius of convergence equal to zero. □

## 4. Some results about the CR operator

We will prove some results about the CR equation that will be needed to construct a $C^\infty$ integral for $L_n$. Consider the space of functions defined in the disc $D(0,R)\subset\mathbb{C}$ by

$$E_R=\{f\in C^\infty(\overline{D(0,R)}\backslash 0);\ f(z)=o(\log^{-q}\frac{1}{|z|})\ \ \forall q>0\}. \tag{4.1}$$

**Lemma 4.1.** *Let $f\in E_R$ and let*

$$g(z)=\int\int_{D(0,R)}\frac{f(\zeta)}{\zeta^2(\zeta-z)}d\xi d\eta, \tag{4.2}$$

*where $\zeta=\xi+i\eta$. Then there is $R_1<R$ such that $zg(z)\in E_{R_1}$.*

*Proof.* Since $f\in E_R$, then it is not difficult to see that $g$ is $C^\infty$ for $z\neq 0$. We need only to show that for a given $q>0$, $zg(z)=o(\log^{-q}\frac{1}{|z|})$. Let

$$D(0,R)=\Delta_1\cup\Delta_2\cup\Delta_3\cup\Delta_4, \tag{4.3}$$

where

$$\begin{aligned}&\Delta_1=D(0,\frac{|z|}{4}),\qquad \Delta_2=D(z,\frac{|z|}{4}),\\ &\Delta_3=\{\zeta:\ \frac{|z|}{4}<|\zeta-z|<\frac{|z|}{4}\log^{q+1}\frac{1}{|z|}\quad\text{and}\quad|\zeta|>\frac{|z|}{4}\},\\ &\Delta_4=\{\zeta:\ \frac{|z|}{4}<|\zeta|<R\quad\text{and}\quad|\zeta-z|>\frac{|z|}{4}\log^{q+1}\frac{1}{|z|}\}.\end{aligned} \tag{4.4}$$

We have

$$\begin{aligned}&|zg(z)|\le|z|I_1+|z|I_2+|z|I_3+|z|I_4\qquad\text{with}\\ &I_j=\int\int_{\Delta_j}\frac{|f(\zeta)|}{|\zeta|^2|\zeta-z|}d\xi d\eta\qquad j=1,2,3,4.\end{aligned} \tag{4.5}$$

To prove the lemma, we need only to show that

$$\lim_{|z|\to0}|z|I_j\log^q\frac{1}{|z|}=0\qquad\text{for } j=1,2,3,4. \tag{4.6}$$

For $\zeta\in\Delta_1$, we have $|\zeta-z|>|z|-|\zeta|>\frac{3}{4}|z|$ and so

$$I_1\le\frac{4}{3|z|}\int\int_{\Delta_1}\frac{|f(\zeta)|}{|\zeta|^2}d\xi d\eta. \tag{4.7}$$

Since $f\in E_R$, then for every $s>0$ there exists $C_s>0$ such that

$$|f(\zeta)|\le C_s\log^{-s}\frac{1}{|\zeta|},\qquad\forall\zeta\in D(0,R). \tag{4.8}$$

Hence,

$$\begin{aligned}I_1&\le\frac{4C_{q+2}}{3|z|}\int\int_{\Delta_1}\frac{d\xi d\eta}{|\zeta|^2\log^{q+2}\frac{1}{|\zeta|}}\\ &\le\frac{8\pi C_{q+2}}{3|z|}\int_0^{|z|/4}\frac{dr}{r\log^{q+2}\frac{1}{r}}\ =\ \frac{8\pi C_{q+2}}{3|z|(q+1)}\log^{-(q+1)}\frac{1}{|z|}\end{aligned} \tag{4.9}$$

and (4.6) holds for $j=1$.

For $\zeta \in \Delta_2$, we have $\dfrac{|z|}{4} < |\zeta| < \dfrac{5}{4}|z|$. Thus,

$$\frac{1}{|\zeta|^2} < \frac{16}{|z|^2} \quad \text{and} \quad \log^{-1}\frac{1}{|\zeta|} < \log^{-1}\frac{4}{5|z|}. \tag{4.10}$$

It follows that

$$\begin{aligned} I_2 &\leq \frac{16C_{q+1}}{|z|^2}\left(\log^{-(q+1)}\frac{4}{5|z|}\right)\int\int_{\Delta_2}\frac{d\xi d\eta}{|\zeta - z|} \\ &\leq \frac{8\pi C_{q+1}}{|z|}\log^{-(q+1)}\frac{4}{5|z|} \end{aligned} \tag{4.11}$$

and (4.6) holds for $j = 2$.

For $\zeta \in \Delta_3$, we have

$$\frac{1}{|\zeta|^2} \leq \frac{16}{|z|^2}. \tag{4.12}$$

We also have

$$|\zeta| \leq |z| + |\zeta - z| \leq |z|(1 + \frac{1}{4}\log^q\frac{1}{|z|}) \leq \sqrt{|z|} \tag{4.13}$$

(we are assuming $|z|$ small). Thus

$$\log^{-1}\frac{1}{|\zeta|} \leq 2\log^{-1}\frac{1}{|z|} \tag{4.14}$$

and

$$\begin{aligned} I_3 &\leq C_{2q+1}\int\int_{\Delta_3}\frac{1}{|\zeta|^2|\zeta - z|}\left(\log^{-(2q+1)}\frac{1}{|\zeta|}\right)d\xi d\eta \\ &\leq \frac{16C_{2q+1}}{|z|^2}2^{2q+1}\left(\log^{-(2q+1)}\frac{1}{|z|}\right)\int\int_{\Delta_3}\frac{d\xi d\eta}{|\zeta - z|}. \end{aligned} \tag{4.15}$$

Using polar coordinates in the last integral, we have

$$\int\int_{\Delta_3}\frac{d\xi d\eta}{|\zeta - z|} \leq 2\pi\int_{|z|/4}^{(|z|/4)\log^q\frac{1}{|z|}} dr \ = \ \frac{|z|}{4}(\log^q\frac{1}{|z|} - 1). \tag{4.16}$$

Therefore,

$$I_3 \leq \frac{8\pi C_{2q+1}2^{2q+1}}{|z|}\left(\log^{-(2q+1)}\frac{1}{|z|}\right)\left(\log^q\frac{1}{|z|} - 1\right) \tag{4.17}$$

and (4.6) holds for $j = 3$.

Finally, for $\zeta \in \Delta_4$, we use

$$|\zeta - z| > \frac{|z|}{4}\log^{q+1}\frac{1}{|z|}$$

to obtain

$$I_4 \leq \frac{4}{|z|}\left(\log^{-(q+1)}\frac{1}{|z|}\right)\int\int_{\Delta_4}\frac{|f(\zeta)|}{|\zeta|^2}d\xi d\eta \leq \frac{4B}{|z|}\left(\log^{-(q+1)}\frac{1}{|z|}\right), \tag{4.18}$$

where

$$B = \int\int_{D(0,R)}\frac{|f(\zeta)|}{|\zeta|^2}d\xi d\eta \ < \ \infty. \tag{4.19}$$

Therefore (4.6) holds for $j = 4$ and the lemma is proved. □

**Theorem 4.1.** *Let $f \in E_R$. Then the CR equation*

$$\frac{\partial w}{\partial \overline{z}} = \frac{f(z)}{z} \tag{4.20}$$

*has a solution $w \in E_R$.*

*Remark* 4.1. Note that, in general, for $f \in E_R$, the function $\dfrac{f}{z} \in L^2$ but $\dfrac{f}{z} \notin L^p$ for any $p > 2$. Hence the classical results about the solvability of the inhomogeneous CR equation cannot be applied here.

*Proof of Theorem* 4.1. The function

$$v(z) = \frac{-1}{\pi} \int\int_{D(0,R)} \frac{f(\zeta)}{\zeta - z} d\xi d\eta \in C^\infty(D(0,R)\backslash 0) \cap C^1(D(0,R)). \tag{4.21}$$

That $v$ is in $C^\infty(D(0,R)\backslash 0) \cap C^\sigma(D(0,R))$ for any $0 < \sigma < 1$ follows from classical theory (see [V], Chapter 1); that $v$ is $C^1$ at 0 follows from a result of [B], Chapter 2. We have

$$\frac{\partial v}{\partial z}(z) = \frac{-1}{\pi} \int\int_{D(0,R)} \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta. \tag{4.22}$$

Let

$$u(z) = v(z) - v(0) - \frac{\partial v}{\partial z}(0) z = \frac{-z^2}{\pi} \int\int_{D(0,R)} \frac{f(\zeta)}{\zeta^2(\zeta - z)} d\xi d\eta. \tag{4.23}$$

The function $u$ solves

$$\frac{\partial u}{\partial \overline{z}} = \frac{\partial v}{\partial \overline{z}} = f(z). \tag{4.24}$$

Therefore, it follows from Lemma 4.1 and from (4.24) that the function

$$w(z) = \frac{u(z)}{z} \in E_R$$

and solves equation (4.20). □

**Lemma 4.2.** *Let $f \in E_R$. The function*

$$Pf(z) = \int\int_{D(0,R)} \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta, \tag{4.25}$$

*where the singular integral is understood in the sense of the Cauchy principal value, satisfies*

$$Pf(z) - Pf(0) \in E_R. \tag{4.26}$$

*Proof.* We know that if $f \in E_R$, then $Pf$ is $C^\infty$ away from 0 (see [V], Chapter 1 ). To prove the lemma, we need only to show that for a given $q > 0$,

$$\lim_{z \to 0} (Pf(z) - Pf(0)) \log^q \frac{1}{|z|} = 0. \tag{4.27}$$

We can rewrite (see [V], page 58)

$$\begin{aligned} Pf(z) - Pf(0) = -z \int\int_{D(0,R)} &\frac{f(\zeta) - f(z)}{(\zeta - z)^2 \zeta} d\xi d\eta \\ &- z \int\int_{D(0,R)} \frac{f(\zeta)}{\zeta^2(\zeta - z)} d\xi d\eta - \pi f(z) \frac{\overline{z}}{z} \end{aligned} \tag{4.28}$$

(in fact, in [V] there is an additional term defined by an integral over the boundary which is equal to zero in our case since $\partial D(0,R)$ is the circle). We then have

$$|Pf(z) - Pf(0)| \leq |z|I_1 + |z|I_2 + \pi|f(z)|, \tag{4.29}$$

where

$$I_1 = \int\int_{D(0,R)} \frac{|f(\zeta) - f(z)|}{|\zeta - z|^2|\zeta|} d\xi d\eta \quad \text{and} \quad I_2 = \int\int_{D(0,R)} \frac{|f(\zeta)|}{|\zeta|^2|\zeta - z|} d\xi d\eta. \tag{4.30}$$

Since $f \in E_R$, to prove the lemma, we need only to show that

$$\lim_{z\to 0} |z| I_k \log^q \frac{1}{|z|} = 0 \quad \text{for} \quad k = 1, 2. \tag{4.31}$$

For $k = 2$, (4.31) holds by Lemma 4.1. To prove it for $k = 1$, notice that since $f \in E_R$, then

$$|f(\zeta) - f(z)| = o(\log^{-q} \frac{1}{|\zeta - z|}) \qquad \forall q > 0 \tag{4.32}$$

uniformly in $z$. Hence for $|z| < \frac{r}{2}$, we have

$$I_1 = \int\int_{D(z,R)} \frac{f(\tau + z) - f(z)}{|\tau|^2|\tau + z|} dsdt \leq \int\int_{D(0,2R)} \frac{|h(\tau, z)|}{|\tau|^2|\tau + z|} dsdt, \tag{4.33}$$

where we have set $\tau = s + it$ and $h(\tau, z) = f(z + \tau) - f(z)$. It follows from (4.32) that $h(., z) \in E_R$ and so (4.31) follows again from Lemma 4.1. □

**Theorem 4.2.** *Let $\mu(z) \in E_R$. The Beltrami equation*

$$\frac{\partial w}{\partial \overline{z}} = \mu(z) \frac{\partial w}{\partial z} \tag{4.34}$$

*has a solution of the form*

$$w(z) = z(1 + K(z)) \tag{4.35}$$

*with $K(z) \in E_R$.*

*Proof.* It follows from classical results that any solution of (4.34) is $C^\infty$ away from 0 (for $R$ small enough) and it follows from [B] (Chapter 3) that equation (4.34) has a $C^1$ solution that is a local diffeomorphism at 0. The local diffeomorphism can be constructed as follows (see [V], Chapter 2). Let

$$w(z) = z + Tf(z), \tag{4.36}$$

with $f$ satisfying the integral equation

$$f(z) - \mu(z)\Pi f(z) = \mu(z), \tag{4.37}$$

where $T$ and $\Pi$ are the integral operators

$$\begin{aligned} Tf(z) &= \frac{-1}{\pi} \int\int_{D(0,R)} \frac{f(\zeta)}{\zeta - z} d\xi d\eta, \\ \Pi f(z) &= \frac{-1}{\pi} \int\int_{D(0,R)} \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta. \end{aligned} \tag{4.38}$$

Furthermore, the function $f$ is obtained as the limit of the sequence $f_n$ defined by

$$f_0 = 0 \quad \text{and} \quad f_{n+1}(z) = \mu(z)\Pi f_n(z) + \mu(z) \quad \text{for } n \geq 0. \tag{4.39}$$

Each $f_n$ is in $E_R$ and so is $f$. The function

$$v(z) = \frac{Tf(z) - Tf(0)}{z} = \frac{-1}{\pi}\int\int_{D(0,R)} \frac{f(\zeta)}{\zeta(\zeta - z)} d\xi d\eta \tag{4.40}$$

solves the equation

$$\frac{\partial v}{\partial \overline{z}} = \frac{f}{z}. \tag{4.41}$$

It follows from Theorem 4.1 that

$$K_1(z) = v(z) - v(0) \ \in \ E_R. \tag{4.42}$$

Hence,

$$w(z) = z + Tf(z) - Tf(0) \tag{4.43}$$

has the desired form. □

## 5. A $C^\infty$ integral

We construct here a $C^\infty$ integral for $L_n$ defined in a tubular neighborhood of the characteristic circle. More precisely, we have the following theorem.

**Theorem 5.1.** *Let $L_n$ be as in* (2.1) *and for $\delta > 0$ let*

$$A_\delta = (-\delta,\ \delta) \times S^1, \quad A_\delta^+ = (0,\ \delta) \times S^1, \quad A_\delta^- = (-\delta,\ 0) \times S^1. \tag{5.1}$$

*Then there is $\delta > 0$ and $h \in C^\infty(A_\delta)$ such that*

(ı ) *$h$ is flat along the circle $\{r = 0\}$;*
(ıı ) *$h :\ A_\delta^\pm \longrightarrow h(A_\delta^\pm)$ is a diffeomorphism; and*
(ııı ) *$L_n h = 0$.*

The rest of the section deals with the proof of this theorem. Let

$$\frac{P(r)}{r^n} + \mu \log|r| + i\theta + \sum_{j=1}^{\infty} f_j(\theta) r^j \tag{5.2}$$

be the series constructed in Section 2, where $P$ is the polynomial of degree $\leq n-1$ given by

$$P(r) = \alpha_{-n} + \alpha_{-n+1} r + \cdots + \alpha_{-1} r^{n-1}. \tag{5.3}$$

Note that

$$P(0) = \frac{-1}{nc_0} = \frac{-1}{n(1+i\beta)} \quad \text{and} \quad Re(P(0)) = \frac{-1}{n(1+\beta^2)} \ < \ 0. \tag{5.4}$$

Let $g(r,\theta) \in C^\infty(A_\delta)$ be such that

$$\frac{\partial^j g}{\partial r^j}(0,\theta) = j! f_j(\theta), \qquad \forall j \in \mathbb{Z}^+. \tag{5.5}$$

Thus the Taylor series of $g$ with respect to $r$ is $\sum f_j(\theta) r^j$. Let

$$m(r,\theta) = \frac{P(r)}{r^n} + \mu \log|r| + i\theta + g(r,\theta). \tag{5.6}$$

The function $m$ is $C^\infty$ in $\mathbb{R} \times \mathbb{R}$ for $r \neq 0$ small, and it satisfies

$$m(r, \theta + 2\pi) = m(r,\theta) + 2\pi \qquad \forall (r,\theta). \tag{5.7}$$

It follows from Proposition 2.1 and from (5.5) that $L_n m$ is flat along $r = 0$. That is,

$$L_n m(r,\theta) = o(r^q) \qquad \forall q > 0. \tag{5.8}$$

Define a function $f \in C^\infty(A_\delta)$ by

$$f(r,\theta) = \exp(\epsilon(r)^n m(r,\theta)), \tag{5.9}$$

where $\epsilon(r) = \dfrac{r}{|r|}$. The function $f$ satisfies

$$|f(r,\theta)| = 0(\exp(\frac{Re(P(0))}{|r|^n})). \tag{5.10}$$

Consequently, $f$ vanishes to infinite order along $r = 0$ since $Re(P(0)) < 0$.

**Lemma 5.1.** *There exists $\delta > 0$ such that the maps*

$$f:\ A_\delta^\pm \longrightarrow f(A_\delta^\pm), \tag{5.11}$$

*defined in* (5.9)*, are diffeomorphisms.*

*Proof.* We will prove the lemma for $A_\delta^+$. We need only to show that there exists $\delta > 0$ for which $f$ is injective in $A_\delta^+$. Consider the equation

$$f(r,\theta) = f(\rho,\phi). \tag{5.12}$$

By equating the real and imaginary parts we obtain

$$\begin{aligned} \frac{P_1(r) + \mu_1 r^n \log r + r^n g_1(r,\theta)}{r^n} &= \frac{P_1(\rho) + \mu_1 \rho^n \log \rho + \rho^n g_1(\rho,\phi)}{\rho^n}, \\ \theta + \frac{P_2(r) + \mu_2 r^n \log r + r^n g_2(r,\theta)}{r^n} &= \phi + \frac{P_2(\rho) + \mu_2 \rho^n \log \rho + \rho^n g_2(\rho,\phi)}{\rho^n} \end{aligned} \tag{5.13}$$

where we have set

$$\mu = \mu_1 + i\mu_2, \quad P = P_1 + iP_2, \quad \text{and} \quad g = g_1 + ig_2.$$

The first equation of (5.13) can be rewritten as

$$\frac{r}{\sqrt[n]{-P_1(r) - \mu_1 r^n \log r - r^n g_1(r,\theta)}} = \frac{\rho}{\sqrt[n]{-P_1(\rho) - \mu_1 \rho^n \log \rho - \rho^n g_1(\rho,\phi)}}. \tag{5.14}$$

We have then, from the implicit function theorem, that for $\delta$ small enough (5.13) has a solution of the form

$$\rho = r(1 + r\alpha(r,\theta)) \quad \text{and} \quad \phi = \theta + r\beta(r,\theta). \tag{5.15}$$

Now, it can be proved that $f$ is a local diffeomorphism in a neighborhood of each point $(r,\theta)$ with $r \neq 0$. This, together with (5.15), imply that the functions $\alpha$ and $\beta$ are identically zero and so $f$ is injective on $A_\delta^+$. A similar argument shows that $f$ is also injective on $A_\delta^-$. □

Since $f$ is flat along $r = 0$, it follows from Lemma 5.1 that there exists $R = R(\delta)$ such that

$$(D(0,R)\backslash 0) \subset f(A_\delta^\pm) \subset \mathbb{C}. \tag{5.16}$$

Let

$$L^\pm = f_* L_n \tag{5.17}$$

be the pushforward to $f(A_\delta^\pm)$ of the vector field $L_n$ via $f$.

**Lemma 5.2.** *There exist a function $A(z) \in E_R$, where $E_R$ is the space of functions defined in* (4.1)*, and a function $B(z)$ with*

$$B \in C^\infty(D(0,R)\backslash 0) \quad \textit{and} \quad B(z)\log^{1/n}\left(\frac{1}{|z|}\right) \textit{ is bounded}$$

*such that the vector field $L^+$ defined by* (5.17) *can be expressed as*

$$L^+ = zA(z)\frac{\partial}{\partial z} - \frac{2i}{\overline{c_0}}\overline{z}(1+B(z))\frac{\partial}{\partial \overline{z}}\,. \tag{5.18}$$

*A similar expression holds for $L^-$.*

*Proof.* Since $z = f(r,\theta)$, then it follows from (5.10) that there exist positive constants $a$ and $b$ such that

$$a\exp(-(\frac{\kappa}{r})^n) \;<\; |z| \;<\; b\exp(-(\frac{\kappa}{r})^n), \tag{5.19}$$

where we have set $\kappa = \sqrt[n]{-P_1(0)}$. Equivalently,

$$\kappa\log^{-\frac{1}{n}}\frac{b}{|z|} \;< r \;<\; \kappa\log^{-\frac{1}{n}}\frac{a}{|z|}\;. \tag{5.20}$$

Let

$$L^+ = X(z)\frac{\partial}{\partial z} + Y(z)\frac{\partial}{\partial \overline{z}} \tag{5.21}$$

where

$$X(z) = (L_n f)(f^{-1}(z)) \quad \text{and} \quad Y(z) = (L_n\overline{f})(f^{-1}(z)). \tag{5.22}$$

Using $f(r,\theta) = \exp m(r,\theta)$, we get

$$L_n f = fL_n m \quad \text{and} \quad L_n\overline{f} = \overline{f}L_n\overline{m}. \tag{5.23}$$

We know that $L_n m$ is flat along $r = 0$, and

$$\begin{aligned} L_n\overline{m} &= (\frac{\partial}{\partial\theta} - ir^{n+1}(c_0 + O(r))\frac{\partial}{\partial r})(\frac{\overline{P}(r)}{r^n} - i\theta + \overline{\mu}\log r + O(r)) \\ &= -i + inc_0\overline{P}(0) + O(r) = \frac{-2i}{\overline{c_0}} + O(r). \end{aligned} \tag{5.24}$$

Hence, it follows from (5.22), (5.23) and (5.24) that

$$X(z) = zA(z) \quad \text{and} \quad Y(z) = \frac{-2i}{\overline{c_0}}\overline{z}(1+B(z)). \tag{5.25}$$

That $A \in E_R$ follows from (5.23), (5.20) and (5.8). That $B$ satisfies the conditions of the lemma follows from (5.22), (5.24), and (5.20). □

We are going to construct a solution to the equation $L_n u = 0$ in $A_\delta^+$ in the form $u = f(r,\theta)(1 + k(r,\theta))$, where $f$ is defined by (5.9) and where $k$ is a $C^\infty$ function vanishing to infinite order along $r = 0$. The function $k$ will be defined as

$$k(r,\theta) = K\circ f(r,\theta) \tag{5.26}$$

where $K(z)$ is a solution of the equation

$$L^+(z(1+K(z))) = 0 \qquad \text{in} \quad D(0,R), \tag{5.27}$$

and where $L^+$ is defined in (5.18).

By using the expression of $L^+$ given in Lemma 5.2, we find that the function $U = \log(1+K)$ must solve the equation

$$\frac{\partial U}{\partial \overline{z}} = \frac{M(z)}{\overline{z}} + \frac{z}{\overline{z}} M(z) \frac{\partial U}{\partial z}, \tag{5.28}$$

where we have set

$$M(z) = \frac{\overline{c_0} A(z)}{2i(1+B(z))} \ \in \ E_R. \tag{5.29}$$

To solve (5.28), we first consider the Beltrami equation

$$\frac{\partial w}{\partial \overline{z}} = \frac{z}{\overline{z}} M(z) \frac{\partial w}{\partial z}. \tag{5.30}$$

Since this equation has a coefficient in $E_R$, then it follows from Theorem 4.2 that it has a solution $w$ of the form

$$w(z) = z(1+s(z)) \quad \text{with} \quad s \in E_R. \tag{5.31}$$

With respect to the new complex variable $w$, equation (5.28) becomes

$$\frac{\partial U}{\partial \overline{z}} = \frac{N(w)}{\overline{w}}, \tag{5.32}$$

where

$$N(w) = \frac{\overline{w} M}{\overline{z}(1-|M|^2)\overline{w_z}}. \tag{5.33}$$

Hence, $N \in E_R$ and by Theorem 4.1, equation (5.32) has a solution $U(w) \in E_R$. The function

$$K(z) = \exp(U(w(z))) - 1 \ \in \ E_R \tag{5.34}$$

solves (5.27) and consequently, the function $k(r,\theta)$ given by (5.26) is flat along $r = 0$ (thanks to (5.20)) and

$$L_n \left( f(r,\theta)(1+k(r,\theta)) \right) = 0 \quad \text{in } A_\delta^+. \tag{5.35}$$

A similar argument gives a solution to the equation $L_n u = 0$ in $A_\delta^-$ of the form $u = f(1+\hat{k})$ with $\hat{k}$ flat along $r = 0$. We define $h$ in $A_\delta$ by

$$h(r,\theta) = \begin{cases} f(r,\theta)(1+k(r,\theta)) & \text{if } r \geq 0, \\ f(r,\theta)(1+\hat{k}(r,\theta)) & \text{if } r \leq 0. \end{cases} \tag{5.36}$$

It follows from the construction of $k$ and $\hat{k}$ that if $\delta$ is small enough, then $h$ satisfies all properties of Theorem 5.1. This completes the proof.

*Remark* 5.1. The integral $h(r,\theta)$ constructed above has the form

$$h(r,\theta) = \exp\left( \epsilon(r)^n \left(\frac{P(r)}{r^n} + \mu \log|r| + i\theta + l(r,\theta)\right)\right) \tag{5.37}$$

with $l \in C^\infty(A_\delta)$ and $l(0,\theta) = 0$. In general, $l$ is not real analytic, even when $L_n$ is real analytic (see Section 3).

## 6. Normalization

We make use of the first integral constructed in the previous section to find a normal form for the vector field $L_n$.

**Theorem 6.1.** *Let $L_n$ be a vector field as in* (2.1)*. Then there exists a unique polynomial $P(r)$ with $Re(P(0)) < 0$ and of degree $\leq n-1$, and there exists a complex number $\mu$ such that $L_n$ is $C^\infty$-conjugate in a ring $A_\delta$ to the vector field*

$$R_n = \frac{\partial}{\partial\theta} - i\frac{r^{n+1}}{rP'(r) - nP(r) + \mu r^n}\frac{\partial}{\partial r}\ , \tag{6.1}$$

*with a $C^\infty$ integral given by*

$$f_n(r,\theta) = \exp\left(\epsilon(r)^n(\frac{P(r)}{r^n} + \mu\log|r| + i\theta)\right), \tag{6.2}$$

*where $\epsilon(r) = \dfrac{r}{|r|}$.*

To prove the theorem, we use the first integral $h(r,\theta)$ given by (5.37). Our aim is to find new coordinates in which the function $l(r,\theta)$ is identically zero. Let

$$A(r) = \frac{P(r)}{r^n} + i\mu\log|r|\ . \tag{6.3}$$

We decompose the functions into their real and imaginary parts:

$$A = A_1 + iA_2, \quad P = P_1 + iP_2, \quad l = l_1 + il_2, \quad \mu = \mu_1 + i\mu_2. \tag{6.4}$$

**Lemma 6.1.** *The equation*

$$A_1(\rho) = A_1(r) + l_1(r,\theta) \tag{6.5}$$

*has a solution $\rho \in C^\infty(A_\delta)$ of the form*

$$\rho = r + r^{n+2}\beta(r,\theta)\ . \tag{6.6}$$

*Proof.* Equation (6.5) can be rewritten as

$$\frac{\rho}{(-P_1(\rho) - \rho^n\mu_1\log|\rho|)^{1/n}} = \frac{r}{(-P_1(r) - r^n\mu_1\log|r| - r^n l_1(r,\theta))^{1/n}}. \tag{6.7}$$

It follows at once from the implicit function theorem that (6.7) has a solution $\rho = r + o(r)$. We write this solution as $\rho = r(1+\alpha(r,\theta))$ and solve for the function $\alpha$. By rewriting (6.5) for the unknown $\alpha$, we get the equation

$$G(r,\theta,\alpha) = 0, \tag{6.8}$$

where $G$ is a $C^\infty$ function defined for $|r| < \delta$, $|\alpha| < \delta$, and $\theta \in S^1$ by

$$G(r,\theta,\alpha) = (1+\alpha)^n\left(P_1(r) + r^n l_1(r,\theta)\right) - P_1(r(1+\alpha)) - \mu_1 r^n\log(1+\alpha). \tag{6.9}$$

Since

$$\frac{\partial G}{\partial\alpha}(0,0,\theta) = nP_1(0) \neq 0, \tag{6.10}$$

and

$$\frac{\partial^j G}{\partial r^j}(0,0,\theta) = 0 \quad \text{for } j = 0,\cdots,n\ , \tag{6.11}$$

the solution $\alpha$ satisfies $\alpha = o(r^n)$. This proves the lemma. □

**Lemma 6.2.** *Let $\rho(r,\theta)$ be a function as in* (6.6). *Then*

$$\log r - \log\rho \in C^\infty(A_\delta) \quad \text{and} \quad \frac{P_2(r)}{r^n} - \frac{P_2(\rho)}{\rho^n} \in C^\infty(A_\delta), \theta 6.12$$

*where $P_2$ is the imaginary part of the polynomial $P$. Furthermore, the functions given in* (6.12) *vanish along $r = 0$.*

*Proof.* For $\rho$ as in (6.6), we have

$$\log r - \log\rho = -\log(1 + r^{n+1}\beta) \tag{6.13}$$

which is clearly $C^\infty$ for $r$ small and vanishes for $r = 0$. We also have

$$\frac{P_2(r)}{r^n} - \frac{P_2(\rho)}{\rho^n} = \frac{(1 + r^{n+1}\beta)^n P_2(r) - P_2(r(1 + r^{n+1}\beta))}{r^n(1 + r^{n+1}\beta)^n}. \tag{6.14}$$

Since

$$(1 + r^{n+1}\beta)^n P_2(r) - P_2(r(1 + r^{n+1}\beta)) = o(r^{n+1}), \tag{6.15}$$

the conclusions follow. □

*Proof of Theorem* 6.1. Let $\rho$ be the solution (6.6) of equation (6.5). With respect to the coordinates $(\rho,\theta)$, the function $h(r,\theta)$ has the expression

$$h(\rho,\theta) = \exp\left[\epsilon(\rho)^n\left(\frac{P_1(\rho)}{\rho^n} + \mu_1\log|\rho| + i(\theta + \frac{P_2(\rho)}{\rho^n} + \mu_2\log|\rho| + s(\rho,\theta))\right)\right], \tag{6.16}$$

where $s$ is given by

$$s(\rho,\theta) = l_2(r,\theta) + \mu_2(\log|r| - \log|\rho|) + \frac{P_2(r)}{r^n} - \frac{P_2(\rho)}{\rho^n}. \tag{6.17}$$

It follows from Lemma 6.2 that $s \in C^\infty(A_\delta)$ and that $s = 0$ along $\rho = 0$. Finally, if we take as a new angle,

$$\phi = \theta + s(\rho,\theta), \tag{6.18}$$

then with respect to the new coordinates $(\rho,\phi)$, the function $h$ has the desired form

$$h(\rho,\theta) = \exp\left(\epsilon(\rho)^n(A(\rho) + i\phi)\right) \tag{6.19}$$

whose annihilator is the vector field $R_n(\rho,\phi)$ given in (6.1). □

For a real analytic vector field $L_n$, the normal form $R_n$ can be achieved under a real analytic diffeomorphism only when the formal integral constructed in Section 2 converges for some $r \neq 0$. Under the assumption that the formal integral converges, the proof of the $C^\omega$-conjugacy is identical to that given above. We state this as the following theorem.

**Theorem 6.2.** *Let $L_n$ be a real analytic vector field as in* (2.1). *Suppose that the corresponding formal solution converges for some $r \neq 0$. Then $L_n$ is $C^\omega$-conjugate in a ring $A_\delta$ to the vector field*

$$R_n = \frac{\partial}{\partial\theta} - i\frac{r^{n+1}}{rP'(r) - nP(r) + \mu r^n}\frac{\partial}{\partial r}\ . \tag{6.20}$$

## 7. The kernel of $R_n$

We determine the structure of the solutions of the homogeneous equation

$$R_n u = 0 \tag{7.1}$$

in the ring $A_\delta = (-\delta,\ \delta) \times S^1$.

**Theorem 7.1.** *Let $f_n$ be the first integral given in* (6.2) *of the vector field $R_n$. A function $u \in C^0(\overline{A_\delta})$ solves* (7.1) *if and only if there exist holomorphic functions $H^\pm$ defined in a neighborhood of $0 \in \mathbb{C}$, with $H^+(0) = H^-(0)$, such that*

$$u(r,\theta) = H^\pm \circ f_n(r,\theta) \qquad \forall (r,\theta) \in \overline{A_\delta^\pm}, \tag{7.2}$$

*where $A_\delta^+ = A_\delta \cap \{r > 0\}$ and $A_\delta^- = A_\delta \cap \{r < 0\}$. Consequently, any $C^0$-solution of* (7.1) *is $C^\infty$.*

*Proof.* The pushforward of $u$ in $A_\delta^+$ via the first integral $f_n$ is a function $H^+$ defined in $f_n(A_\delta^+)$ that satisfies the CR equation $H_{\overline{z}}^+ = 0$. Hence, $H^+$ is a bounded holomorphic function defined in a neighborhood of $0 \in \mathbb{C}$. Therefore, $u = H^+ \circ f_n$ in $A_\delta^+$. A similar result holds in $A_\delta^-$. That $H^+(0) = H^-(0)$ follows from the continuity of $u$ and that $u$ is $C^\infty$ on $r = 0$ follows from the flatness of $f_n$ along $r = 0$. □

*Remark* 7.1. Theorem 7.1 does not have a local counterpart version. For every $p \in \Sigma$ there exist $C^0$ solutions of $L_n u = 0$ defined in a neighborhood of $p$ that are not $C^\infty$. For example, for a given branch of the logarithm, the function $(x+ix^2t)^{3/2}$ is not $C^\infty$ in a neighborhood of 0 and it satisfies the equation

$$(\frac{\partial}{\partial t} - i\frac{x^2}{1+2ixt}\frac{\partial}{\partial x})u = 0$$

(we refer to [T1] and [T2] for the local solvabilty of vector fields).

The next result describes the distribution solutions of (7.1) that are supported by the characteristic circle $r = 0$. The analogue question for the vector field $L_0$ is treated in [BhM2].

**Theorem 7.2.** *Let $u \in \mathcal{D}'(A_\delta)$ with supp$(u) \subset \{r = 0\}$. If $u$ solves* (7.1)*, then there exist constants $c_0, \cdots, c_{n-1}$ in $\mathbb{C}$ such that*

$$\langle u, \phi\rangle = \sum_{j=0}^{n-1} c_j \int_0^{2\pi} \frac{\partial^j \phi}{\partial r^j}(0,\theta)d\theta \qquad \forall \phi \in \mathcal{D}(A_\delta). \tag{7.3}$$

*Proof.* The transpose of $R_n$ is the operator

$$R_n^* = -\frac{\partial}{\partial\theta} + ir^{n+1}Q(r)\frac{\partial}{\partial r} + i(r^{n+1}Q(r))_r, \tag{7.4}$$

where

$$Q(r) = \frac{1}{rP'(r) - nP(r) + \mu r^n}.$$

First, we verify that a distribution $u$ given by (7.3) solves equation (7.1). For $j = 0, \cdots, n-1$, let $u_j \in \mathcal{D}'(A_\delta)$ be defined by

$$\langle u_j, \phi\rangle = \int_0^{2\pi} \frac{\partial^j \phi}{\partial r^j}(0,\theta)d\theta \qquad \forall \phi \in \mathcal{D}(A_\delta). \tag{7.5}$$

For $\phi \in \mathcal{D}(A_\delta)$, we write

$$\phi(r,\theta) = \sum_{k=0}^{n-1} l_k(\theta) r^k + O(r^n), \quad l_k(\theta) \in C^\infty(S^1). \tag{7.6}$$

It follows that

$$R_n^* \phi = -\sum_{k=0}^{n-1} l_k'(\theta) r^k + O(r^n). \tag{7.7}$$

Therefore,

$$\frac{\partial^j R_n^* \phi}{\partial r^j}(0,\theta) = -j! l_j'(\theta), \tag{7.8}$$

and

$$\langle R_n u_j, \phi \rangle = \langle u_j, R_n^* \phi \rangle = \int_0^{2\pi} \frac{\partial^j R_n^* \phi}{\partial r^j}(0,\theta) d\theta = -j! \int_0^{2\pi} l_j'(\theta) d\theta = 0. \tag{7.9}$$

Hence, $u_j$ solves (7.1) and so does any linear combination given by (7.3).

Let $u \in \mathcal{D}'(A_\delta)$ be a solution of (7.1) and supp $(u) \subset \{r = 0\}$. Suppose that $u$ has a transverse order $m$. Since $R_n$ is elliptic in the tangential direction along $r = 0$, then there exist $a_0(\theta), \cdots, a_m(\theta) \in C^\infty(S^1)$ such that

$$\langle u, \phi \rangle = \sum_{k=0}^{m} \int_0^{2\pi} a_k(\theta) \frac{\partial^k \phi}{\partial r^k}(0,\theta) d\theta. \tag{7.10}$$

We prove that the order $m$ must satisfy $m \leq n-1$. By contradiction, suppose that $m \geq n$. Then $a_m \neq 0$ and there exists $p \in \mathbb{Z}$ such that

$$\int_0^{2\pi} a_m(\theta) \mathrm{e}^{ip\theta} d\theta \neq 0. \tag{7.11}$$

If $p \neq 0$, we let $\phi \in \mathcal{D}(A_\delta)$ be of the form

$$\phi(r,\theta) = \mathrm{e}^{ip\theta} r^m + o(r^m). \tag{7.12}$$

Then

$$R_n^* \phi = -ip\mathrm{e}^{ip\theta} r^m + o(r^m) \tag{7.13}$$

and

$$\begin{aligned} &\frac{\partial^j R_n^* \phi}{\partial r^j}(0,\theta) = 0 \qquad \text{if } j = 0, \ldots, m-1, \\ &\frac{\partial^m R_n^* \phi}{\partial r^m}(0,\theta) = -m! ip\mathrm{e}^{ip\theta}. \end{aligned} \tag{7.14}$$

It follows from (7.10), (7.11) and (7.14) that

$$\langle R_n u, \phi \rangle = \langle u, R_n^* \phi \rangle = \int_0^{2\pi} a_m(\theta)(-m! ip\mathrm{e}^{ip\theta}) d\theta \neq 0. \tag{7.15}$$

This contradicts the hypothesis $R_n u = 0$ and shows that $m \leq n-1$ when $p \neq 0$. In the case $p = 0$, we consider $\phi \in \mathcal{D}(A_\delta)$ independent of $\theta$ and given by

$$\phi(r,\theta) = g(r) = r^{m-n} + o(r^{m-n}) \tag{7.16}$$

with $g \in \mathcal{D}((-\delta,\ \delta))$. We have (by using (7.4)) that

$$R_n^* \phi(r,\theta) = i\frac{d}{dr}(r^{n+1} Q(r) g(r)) = i(m+1)Q(0) r^m + o(r^m). \tag{7.17}$$

Consequently,

$$\begin{aligned}\frac{\partial^j R_n^* g}{\partial r^j}(0,\theta) &= 0 \qquad \text{if } j = 0,\ldots,m-1,\\ \frac{\partial^m R_n^* \phi}{\partial r^m}(0,\theta) &= i(m+1)!Q(0) \neq 0.\end{aligned} \tag{7.18}$$

A similar argument shows that in this case, we also have $\langle R_n u, g\rangle \neq 0$. This shows that the order of a distribution solution supported by the characteristic circle needs to be $\leq n-1$.

Next, we prove that the coefficients $a_0(\theta), \cdots, a_m(\theta)$ given in (7.10) are constants. For $\phi \in \mathcal{D}(A_\delta)$, we have

$$0 = \langle R_n u, \phi\rangle = \sum_{j=0}^{m} \int_0^{2\pi} a_j(\theta) \frac{\partial^j R_n^* \phi}{\partial r^j}(0,\theta) d\theta. \tag{7.19}$$

We write

$$\phi(r,\theta) = \sum_{k=0}^{m} l_k(\theta) r^k + o(r^m). \tag{7.20}$$

Since $m < n$, it follows from (7.4) and (7.20) that for $j = 0, \cdots, m$,

$$\frac{\partial^j R_n^* \phi}{\partial r^j}(0,\theta) = j! l_j'(\theta). \tag{7.21}$$

For a given $k \leq m$, let $\phi_k \in \mathcal{D}'(A_\delta)$ be such that

$$\phi_k(r,\theta) = l_k(\theta) r^k + o(r^m). \tag{7.22}$$

Equation (7.19) together with (7.21) gives

$$0 = \langle R_n u, \phi_k\rangle = k! \int_0^{2\pi} a_k(\theta) l_k'(\theta) d\theta. \tag{7.23}$$

If $a_k$ were not constant, then there would be $p \in \mathbb{Z}$ with $p \neq 0$ such that

$$\int_0^{2\pi} a_k(\theta) \mathrm{e}^{ip\theta} d\theta \neq 0, \tag{7.24}$$

and in this case if we select $f_k(\theta) = \mathrm{e}^{ip\theta}$, then (7.23) will be violated. This shows that $a_0, \cdots, a_m$ are constants. ☐

## 8. A degenerate Beltrami equation

The Beltrami equation $w_{\overline{z}} = \mu(z) w_z$ has been studied in the elliptic case $|\mu(z)| \leq K < 1$ for all $z$ in a domain of $\mathbb{C}$ (see [V]). However, very little is known when $\mu(z)$ is not uniformly bounded away from 1. In this section, we consider this degenerate situation and show that it can be understood in terms of the vector field $L_n$ with $n = 0$.

We start with a vector field $V$ defined in a disc $D(0,\delta) \subset \mathbb{C}$ by

$$V = A(z)\frac{\partial}{\partial z} + B(z)\frac{\partial}{\partial \overline{z}}, \tag{8.1}$$

where $A, B \in C^l(D(0,R))$ satisfy

$$|A(z)| = O(|z|^m) \quad \text{and} \quad |B(z)| = O(|z|^n) \tag{8.2}$$

with $m, n \in \mathbb{Z}^+$ such that

$$n \leq m < l. \tag{8.3}$$

Assume that there exist constants $a, b > 0$ such that

$$a|z|^{2n} \leq |B(z)|^2 - |A(z)|^2 \leq b|z|^{2n}. \tag{8.4}$$

The equation $Vw = 0$ is equivalent to the Beltrami equation

$$w_{\overline{z}} = \mu(z) w_z, \tag{8.5}$$

with

$$\mu(z) = -\frac{A(z)}{B(z)}. \tag{8.6}$$

It follows from hypothesis (8.4) that

$$|\mu(z)| < 1 \qquad \text{for } z \neq 0. \tag{8.7}$$

Hence, equation (8.5) is elliptic in a neighborhood of each point $z \neq 0$, but $\limsup_{z \to 0} |\mu(z)|$ might be equal to 1. We will show that this degenerate Beltrami equation has a solution which is a local homeomorphism at 0.

**Theorem 8.1.** *Let $\mu(z)$ be given by* (8.6) *with $A$ and $B$ satisfying* (8.4)*. Then there exist $\delta > 0$, $\sigma > 0$ and a function*

$$w \in C^{l+1}(D(0,R) \backslash 0) \cap C^{\sigma}(D(0,R)) \tag{8.8}$$

*such that $w$ solves the Beltrami equation* (8.5) *and*

$$w : \ D(0,R) \longrightarrow w(D(0,R)) \tag{8.9}$$

*is a homeomorphism.*

*Proof.* First, consider the case $m > n$. It follows from the hypotheses that

$$\mu(z) \in C^{l}(D(0,R) \backslash 0) \cap C^{m-n-1+\sigma}(D(0,R)) \tag{8.10}$$

for any $0 < \sigma < 1$ and that $\mu(0) = 0$. This is a classical Beltrami equation and a diffeomorphic solution $w$ can be found in $D(0,R)$. With $w$ of class $C^{l+1}$ away from 0, and of class $C^{m-n+\sigma}$ at 0 (see [V]).

Next, in the case $m = n$, in which there is an effective degeneracy, let

$$\begin{aligned} A(z) &= A_n(z) + O(|z|^{n+1}), \\ B(z) &= B_n(z) + O(|z|^{n+1}) \end{aligned} \tag{8.11}$$

with $A_n$ and $B_n$ homogeneous polynomials in $z$ and $\overline{z}$ of degree $n$. We use polar coordinates $z = r\mathrm{e}^{i\theta}$ to express (8.4) as

$$a \leq |B(\mathrm{e}^{i\theta})|^2 - |A(\mathrm{e}^{i\theta})|^2 \leq b \tag{8.12}$$

and the vector field $V$ as

$$V = \frac{i}{2} r^{n-1} \left(B_n(\mathrm{e}^{i\theta})\mathrm{e}^{i\theta} - A_n(\mathrm{e}^{i\theta})\mathrm{e}^{-i\theta} + O(r)\right) \left(\frac{\partial}{\partial\theta} - ir(a(\theta) + O(r))\frac{\partial}{\partial r}\right), \tag{8.13}$$

where

$$a(\theta) = \frac{B_n(\mathrm{e}^{i\theta})\mathrm{e}^{i\theta} + A_n(\mathrm{e}^{i\theta})\mathrm{e}^{-i\theta}}{B_n(\mathrm{e}^{i\theta})\mathrm{e}^{i\theta} - A_n(\mathrm{e}^{i\theta})\mathrm{e}^{-i\theta}}. \tag{8.14}$$

It follows from (8.12) that the real part of $a(\theta)$ is nowhere zero. We can therefore assume that $Re(a) > 0$ so that

$$\lambda = \frac{1}{2\pi}\int_0^{2\pi} a(\theta)d\theta \in \mathbb{R}^+ + i\mathbb{R}. \tag{8.15}$$

It follows then from [M2] that there exists $\delta > 0$ such that the equation $Vu = 0$ has a solution of the form

$$u(r,\theta) = r^{1/\lambda}\mathrm{e}^{i\theta}B(r,\theta), \tag{8.16}$$

with

$$B \in C^{l+1}(A_\delta\backslash\{r=0\}) \cup C^0(A_\delta) \tag{8.17}$$

and $B(0,\theta) \neq 0$ for every $\theta$. Hence, for $\delta$ small enough $u$ is a homeomorphism from $A_\delta^+$ onto its image $u(A_\delta^+)$. The expression of the function $u$ in terms of the variable $z = r\mathrm{e}^{i\theta}$ is

$$w(z) = \frac{z}{|z|}|z|^{1/\lambda}\hat{B}(Z) \tag{8.18}$$

with $\hat{B}(z) = B(u^{-1}(z))$. Thus

$$w \in C^l(D(0,\epsilon)\backslash 0) \cap C^\sigma(D(0,\epsilon)) \tag{8.19}$$

for any positive number $\sigma$ satisfying

$$0 < \sigma < Re(\frac{1}{\lambda}). \tag{8.20}$$

Furthermore, $w$ is a homeomorphism and satisfies equation (8.5). $\square$

As a consequence of Theorem 8.1 we get the following factorization result.

**Theorem 8.2.** *If $u$ is a $C^0$-solution of* (8.5) *defined near $0 \in \mathbb{C}$, then there exists a holomorphic $H$ such that $u = H \circ w$, where $w$ is the homeomorphic solution of* (8.5) *as in Theorem* 8.1.

*Remark* 8.1. The following question (motivated by geometric considerations) is considered in [W] (page 52). Given $A(x,y)$ and $B(x,y)$ real analytic functions defined near $0 \in \mathbb{R}^2$ such that

$$\left|\frac{A(x,y)}{B(x,y)}\right| \leq K < 1 \qquad \text{for} \quad x^2 + y^2 \leq R^2,$$

the question is to determine whether the Beltrami equation $w_{\overline{z}} = (A/B)w_z$ has a meromorphic solution. That is, a solution of the form

$$w(x,y) = \frac{f(x,y)}{g(x,y)}$$

with $w$ a local homeomorphism at 0 and $f$ and $g$ real analytic. In view of Theorem 8.1 and its proof, in general, there are no nontrivial meromorphic solutions to such Beltrami equations. Indeed, a necessary condition for the existence of a meromorphic solution is that the invariant $\lambda$ (see (8.15)) of the associated vector field $V$ (as in (8.13)) must be in $\mathbb{Z}^+$. This follows from the fact that if the solution is meromorphic, then its expression in polar coordinates $(r,\theta)$ would be a real analytic integral of $V$ and thus $\lambda \in \mathbb{Z}^+$ (see [M1] and [M2]). However, for given real analytic functions $A$, $B$, the associated invariant $\lambda$ is not necessarily in $\mathbb{Z}$. In fact, even when $\lambda \in \mathbb{Z}^+$, there are vector fields without nontrivial $C^\omega$ solutions (see [CG]).

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